2 Measures

The basic idea behind integration theory via measures may be roughly described as follows: Given a space (set) we want to associate "sizes" to "pieces" of the space. To do this we first have to make precise what we mean by a "piece", i.e., what subsets we admit as "pieces". This is the purpose of the concept of a σ-algebra and a measurable space. Given that we know what a piece is, we want to assign a number to it, its "size", in such a way that sizes add up appropriately when we join pieces. This is provided by the concept of a measure. Then, we can declare the integral for the characteristic function on a piece to be the size of the piece. Approximating more arbitrary functions by linear combinations of characteristic functions for pieces then yields a general notion of integral.

2.1 σ-Algebras and Measurable Spaces

Definition 2.1 (Boolean Algebra)**.** Let A be a set equipped with three operations: $\wedge : A \times A \to A$, $\vee : A \times A \to A$ and $\neg : A \to A$ and two special elements $0, 1 \in A$. Suppose these satisfy the following properties:

- $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ and $(x \vee y) \vee z = x \vee (y \vee z) \quad \forall x, y, z \in A$. (associativity)
- $x \wedge y = y \wedge x$ and $x \vee y = y \vee x \quad \forall x, y \in A$. (commutativity)
- $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \quad \forall x, y, z \in A$. (distributivity)
- $x \wedge (x \vee y) = x$ and $x \vee (x \wedge y) = x \quad \forall x, y \in A$. (absorption)
- $x \wedge \neg x = 0$ and $x \vee \neg x = 1 \quad \forall x \in A$. (complement)

Then, A is called a *Boolean algebra.*

Proposition 2.2. *Let A be a Boolean algebra. Then, the following properties hold:*

$$x \wedge x = x,\ x \vee x = x,\ x \wedge 0 = 0,\ x \wedge 1 = x,\ x \vee 0 = x,\ x \vee 1 = 1 \quad \forall x \in A.$$

Proof. **Exercise.** □

Exercise 9. Show that the set with two elements $0, 1$ forms a Boolean algebra. This is important in logic, where 0 stands for "false" and 1 for "true".

Exercise 10. Let S be a set. Show that the set $\mathfrak{P}(S)$ of subsets of S forms a Boolean algebra, where $\vee = \cup$ is the union, $\wedge = \cap$ is the intersection and $\neg$ is the complement of sets.

Definition 2.3 (Algebra of sets)**.** Let S be a set. A subset $\mathcal{M}$ of the set $\mathfrak{P}(S)$ of subsets of S is called an *algebra* of sets iff it is a Boolean subalgebra of $\mathfrak{P}(S)$.

Proposition 2.4. *Let S be a set and $\mathcal{M}$ a subset of the set $\mathfrak{P}(S)$ of subsets of S. Then $\mathcal{M}$ is an algebra of sets iff it contains the empty set and is closed under complements, finite unions, and finite intersections.*

Proof. Immediate. □

Exercise 11. Show that the above proposition remains true if we erase either the requirement for closedness under finite unions or the requirement for closedness under finite intersections.

Definition 2.5. Let S be a set and $\mathcal{M}$ an algebra of subsets of S. We call $\mathcal{M}$ a *σ-algebra* of sets iff it is closed under countable unions and countable intersections.

Exercise 12. Show that the above definition remains unchanged if we remove either the requirement for closedness under countable unions or closedness under countable intersections.

Definition 2.6. Let S be a set and $\mathcal{B}$ a subset of the set $\mathfrak{P}(S)$ of subsets of S. Then, the smallest σ-algebra $\mathcal{M}$ on S containing $\mathcal{B}$ is called the σ-algebra *generated* by $\mathcal{B}$.

Exercise 13. Justify the above definition by showing that the smallest σ-algebra in the sense of the definition always exists.

Definition 2.7. Let S be a set and $\mathcal{B}$ a subset of $\mathfrak{P}(S)$. Then, $\mathcal{B}$ is called *monotone* iff it satisfies the following properties:

- Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of elements of $\mathcal{B}$ such that $A_n \subseteq A_{n+1}$. Then, $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{B}$.
- Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of elements of $\mathcal{B}$ such that $A_n \supseteq A_{n+1}$. Then, $\bigcap_{n\in\mathbb{N}} A_n \in \mathcal{B}$.

Proposition 2.8. *1. A σ-algebra is monotone. 2. An algebra that is monotone is a σ-algebra.*

Proof. **Exercise.** □

Proposition 2.9 (Monotone Class Theorem)**.** *Let S be a set and $\mathcal{N}$ an algebra of subsets of S. Then, the smallest set $\mathcal{M}$ of subsets of S which contains $\mathcal{N}$ and is monotone is the σ-algebra generated by $\mathcal{N}$.*

Proof. For each $A \in \mathcal{M}$ and consider

$$\mathcal{M}_A := \{B \in \mathcal{M} : A \cap B \in \mathcal{M}, A \cap \neg B \in \mathcal{M}, \neg A \cap B \in \mathcal{M}\}.$$

It is easy to see that $\mathcal{M}_A$ is monotone. [**Exercise.**Show this!] Furthermore, if $A \in \mathcal{N}$, then $\mathcal{N} \subseteq \mathcal{M}_A$ since $\mathcal{N}$ is an algebra. So in this case $\mathcal{M} \subseteq \mathcal{M}_A$ by minimality of $\mathcal{M}$ and consequently $\mathcal{M} = \mathcal{M}_A$. Thus, for $B \in \mathcal{M}$ we have $B \in \mathcal{M}_A$ and hence $A \in \mathcal{M}_B$ if $A \in \mathcal{N}$. So, $\mathcal{N} \subseteq \mathcal{M}_B$ and by minimality we conclude $\mathcal{M} = \mathcal{M}_B$ for any $B \in \mathcal{M}$. But this means that $\mathcal{M}$ is an algebra. Thus, by Proposition 2.8.2, $\mathcal{M}$ is a σ-algebra. Furthermore, by minimality and Proposition 2.8.1, it is the σ-algebra generated by $\mathcal{N}$. □

Definition 2.10. Let S be a set and $\mathcal{M}$ a σ-algebra of subsets of S. Then, we call the pair $(S, \mathcal{M})$ a *measurable space* and the elements of $\mathcal{M}$ *measurable sets*.

Definition 2.11. Let S be a measurable space and U a subset of S. Then, the σ-algebra on S intersected with U is called the *induced* σ-algebra on U.

Definition 2.12. Let S be a topological space. Then, the σ-algebra generated by the topology of S is called the algebra of *Borel* sets. Its elements are called *Borel measurable*.

2.2 Measurable Functions

As we see the concept of a measurable space is very similar to the concept of a topological space. Both are based on a set of subsets closed under certain operations. We can push this analogy further and consider the analog of a continuous function: a measurable function.

Definition 2.13. Let S, T be measurable spaces. Then a map $f : S \to T$ is called *measurable* iff the preimage of every measurable set of T is a measurable set of S. If either T or S or T and S are topological spaces instead we call f measurable iff it is measurable with respect to the generated σ-algebra(s) of Borel sets.

Proposition 2.14. *Let S, T, U be measurable spaces, $f : S \to T$ and $g : T \to U$ measurable. Then, $g \circ f : S \to U$ is measurable.*

Proof. Immediate. □

Proposition 2.15. *Let S be a measurable space, T a topological space and $f : S \to T$. Then, f is measurable iff the preimage of every open set is measurable. Also, f is measurable iff the preimage of every closed set is measurable.*

Proof. **Exercise.** □

Corollary 2.16. *Let S and T be topological spaces and $f : S \to T$ a continuous map. Then, f is measurable.*

Proposition 2.17. *Let S be a measurable space, T and U topological spaces, $f : S \to T \times U$. Denote by $f_T : S \to T$ and $f_U : S \to U$ the component functions. If the product $f : S \to T \times U$ is measurable, then both f_T and f_U are measurable. Conversely, if T and U are second-countable and f_T and f_U are measurable, then f is measurable.*

Proof. First suppose that f is measurable. Then, $f_T = p_T \circ f$, where p_T is the projection $T \times U \to T$. Since p_T is continuous, it is measurable by Corollary 2.16 and the composition f_T is measurable by Proposition 2.14. In the same way it follows that f_U is measurable.

Conversely, suppose now that f_T and f_U are measurable. If $V \subseteq T$ and $W \subseteq U$ are open sets, then $f_T^{-1}(V)$ and $f_U^{-1}(W)$ are measurable in S and so is their intersection $f^{-1}(V \times W) = f_T^{-1}(V) \cap f_U^{-1}(W)$. Since T and U are second-countable, every open set in $T \times U$ can be written as a countable union of products of open sets in T and U [**Exercise.**Show this!]. But the preimage of such a countable union in S under f^{-1} can be written as a countable union of preimages. Since these are measurable, their countable union is also measurable. It follows then from Proposition 2.15 that f is measurable. □

In the following $\mathbb{K}$ denotes either the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$.

Proposition 2.18. *Let S be a measurable space, $f, g : S \to \mathbb{K}$ measurable and $\lambda \in \mathbb{K}$. Then:*

- *$|f| : x \mapsto |f(x)|$ is measurable.*
- *$f + g : x \mapsto f(x) + g(x)$ is measurable.*
- *$\lambda f : x \mapsto \lambda f(x)$ is measurable.*
- *$fg : x \mapsto f(x)g(x)$ is measurable.*

Proof. **Exercise.** □

This shows in particular that measurable functions with values in $\mathbb{R}$ or $\mathbb{C}$ form an algebra. Another important property of the set of measurable maps is its closedness under pointwise limits. This can be formulated for the more general case when the values are taken in a metric space.

Theorem 2.19 (adapted from S. Lang)**.** *Let S be a measurable space and T a metric space. Suppose $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of measurable functions $f_n : S \to T$ which converges pointwise to the function $f : S \to T$. Then, f is measurable.*

Proof. Let U be an open set in T. Suppose $x \in f^{-1}(U)$. Since $\{f_n(x)\}_{n\in\mathbb{N}}$ converges to $f(x)$ there exists $m \in \mathbb{N}$ such that $x \in f_n^{-1}(U)$ for all $n > m$. In particular, $x \in \bigcup_{n=k}^{\infty} f_n^{-1}(U)$ for any $k \in \mathbb{N}$ and so also $x \in \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} f_n^{-1}(U)$. Since this is true for any $x \in f^{-1}(U)$ we get

$$f^{-1}(U) \subseteq \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} f_n^{-1}(U).$$

Consider now for all $l \in \mathbb{N}$ the open sets

$$U_l := \{x \in U : d(x,y) > 1/l \ \forall y \notin U\}.$$

Then, $U = \bigcup_{l=1}^{\infty} U_l$ and applying the above reasoning to each U_l we get,

$$f^{-1}(U) \subseteq \bigcup_{l=1}^{\infty}\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} f_n^{-1}(U_l).$$

Suppose now that $x \notin f^{-1}(U)$ and fix $l \in \mathbb{N}$. Since $B_{1/l}(f(x)) \cap U_l = \emptyset$ there exists $m \in \mathbb{N}$ such that $x \notin f_n^{-1}(U_l)$ for all $n > m$. In particular, $x \notin \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} f_n^{-1}(U_l)$. Since this is true for any $l \in \mathbb{N}$ we get $x \notin \bigcup_{l=1}^{\infty}\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} f_n^{-1}(U_l)$. Since this is true for any $x \notin f^{-1}(U)$ we get, combining with the above result,

$$f^{-1}(U) = \bigcup_{l=1}^{\infty}\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} f_n^{-1}(U_l).$$

Since f_n is measurable for all $n \in \mathbb{N}$ the right hand side is measurable. We have thus shown that preimages of open sets are measurable. By Proposition 2.15 this is sufficient for f to be measurable. □

Definition 2.20. Let S be a measurable space. A map $f : S \to \mathbb{K}$ is called a *simple map* iff it is measurable and takes only finitely many values.

Proposition 2.21. *Let S be a measurable space and $f : S \to \mathbb{K}$ a map that takes only finitely many values. Then f is a simple map (i.e., is measurable) iff the preimage of each of the values of f is measurable.*

Proof. **Exercise.** □

Proposition 2.22. *The simple functions with values in $\mathbb{K}$ form a subalgebra of the algebra of measurable functions with values in $\mathbb{K}$.*

Proof. **Exercise.** □

Theorem 2.23 (adapted from S. Lang)**.** *Let S be a measurable space and $f : S \to \mathbb{K}$ measurable. Then, f is the pointwise limit of a sequence of simple maps. If, moreover, f takes values in $\mathbb{R}_0^+$, then the sequence can be chosen to increase monotonically.*

Proof. Consider first the case $\mathbb{K} = \mathbb{R}$. Fix $n \in \mathbb{N}$. For each $k \in \{1, \ldots, 2^{n+1}n\}$ define the interval $I_k := [-n + \frac{k-1}{2^n}, -n + \frac{k}{2^n})$. Also, define $I_0 := (-\infty, -n)$ and $I_{2^{n+1}n+1} := [n, \infty)$. Notice that $\mathbb{R}$ is the disjoint union of the measurable intervals I_k for $k \in \{0, \ldots, 2^{n+1}n + 1\}$. Now set $X_k := f^{-1}(I_k)$ for all $k \in \{0, \ldots, 2^{n+1}n + 1\}$. Since the intervals I_k are measurable so are the sets X_k. Define the function $f_n : X \to \mathbb{R}$ by $f_n(X_k) := -n + \frac{k-1}{2^n}$ for all $k \in \{1, \ldots, 2^{n+1}n+1\}$ and $f_n(X_0) := -n$. It is easy to see that $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of simple functions that converge pointwise to f. [**Exercise.**Show this!] Moreover, if f takes values in $\mathbb{R}_0^+$ only, the sequence is monotonically increasing. [**Exercise.**Show this!] To treat the case $\mathbb{K} = \mathbb{C}$ we decompose f into its real and imaginary part. The sum of simple sequences for each part is again a simple sequence. □

2.3 Positive Measures

Definition 2.24. Let $\{a_n\}_{n\in\mathbb{N}}$ be a monotonously increasing sequence of real numbers. Then we say that $\lim_{n\to\infty} a_n = \infty$ iff for any $a \in \mathbb{R}$ there exists $m \in \mathbb{N}$ such that $a_n > a$ for all $n > m$.

Definition 2.25 (Positive Measure)**.** Let S be a set with an algebra $\mathcal{M}$ of subsets. Then, a map $\mu : \mathcal{M} \to [0, \infty]$ is called a *(positive) measure* iff it is countably additive, i.e., satisfies the following properties:

- $\mu(\emptyset) = 0$.
- Let $\{U_n\}_{n\in\mathbb{N}}$ be a sequence of elements of $\mathcal{M}$ such that $U_n \cap U_m = \emptyset$ if $n \neq m$ and such that $\bigcup_{n\in\mathbb{N}} U_n \in \mathcal{M}$. Then,
$$\mu\left(\bigcup_{n\in\mathbb{N}} U_n\right) = \sum_{n\in\mathbb{N}} \mu\left(U_n\right).$$

If $U \in \mathcal{M}$, then $\mu(U)$ is called its *measure*. Moreover, a measurable space S with σ-algebra $\mathcal{M}$ and positive measure $\mu : \mathcal{M} \to [0, \infty]$ is called a *measure space*.

We shall mostly be interested in the case where $\mathcal{M}$ actually is a σ-algebra. However, it will turn out convenient to keep the definition more general when we consider constructing measures.

Proposition 2.26. *Let S be a set, $\mathcal{M}$ an algebra of subsets of S and $\mu : \mathcal{M} \to [0, \infty]$ a measure. Then, the following properties hold:*

- *Let $A, B \in \mathcal{M}$ and $A \subseteq B$. Then, $\mu(A) \leq \mu(B)$.*
- *Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of elements of $\mathcal{M}$ such that $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{M}$. Then,*
$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leq \sum_{n\in\mathbb{N}} \mu(A_n).$$

- *Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of elements of $\mathcal{M}$ such that $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{M}$. Then,*

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

- *Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of elements of $\mathcal{M}$ such that $A_n \supseteq A_{n+1}$ for all $n \in \mathbb{N}$ and $\bigcap_{n\in\mathbb{N}} A_n \in \mathcal{M}$. If furthermore, $\mu(A_n) < \infty$ for some $n \in \mathbb{N}$ then,*

$$\mu\left(\bigcap_{n\in\mathbb{N}} A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

Proof. **Exercise.** □

Exercise 14. Check whether the following examples are measures.

- Let S be a set and consider the σ-algebra of all subsets of S. If $A \subseteq S$ is finite define $\mu(A)$ to be its number of elements. If $A \subseteq S$ is infinite define $\mu(A) = \infty$. μ is called the *counting measure*.
- Let S be a set and consider the σ-algebra of all subsets of S. If $A \subseteq S$ is finite define $\mu(A) = 0$. If $A \subseteq S$ is infinite define $\mu(A) = \infty$.
- Let S be a set and consider the σ-algebra of all subsets of S. If $A \subseteq S$ is countable define $\mu(A) = 0$. If $A \subseteq S$ is not countable define $\mu(A) = \infty$.
- Let S be a set and consider the σ-algebra of all subsets of S. Let $x \in S$. For $A \subseteq S$ define $\mu(A) = 1$ if $x \in A$ and $\mu(A) = 0$ otherwise. μ is called the *Dirac measure* with respect to x.

Definition 2.27. Let S be a measure space and $A \subseteq S$ a measurable subset. We say that A is *σ-finite* iff it is equal to some countable union of measurable sets with finite measure. We say that a measure is *finite* respectively *σ-finite* iff the measure space is finite respectively σ-finite with respect to the measure.

Exercise 15. Which of the examples of measures above are σ-finite?

Definition 2.28. Let $(S, \mathcal{M}, \mu)$ be a measure space. If every subset of any set of measure 0 is measurable, then we call $(S, \mathcal{M}, \mu)$ *complete*.

Proposition 2.29. *Let $(S, \mathcal{M}, \mu)$ be a measure space. Then, there exists a unique smallest σ-algebra $\mathcal{M}^*$ that contains $\mathcal{M}$ and such that $(S, \mathcal{M}^*, \mu)$ is complete. $(S, \mathcal{M}^*, \mu)$ is called the* completion *of $(S, \mathcal{M}, \mu)$. Moreover, the element of $\mathcal{M}^*$ are precisely the sets of the form $A \cup N$, where $A \in \mathcal{M}$ and N is a subset of a set of measure 0 in $\mathcal{M}$.*

Proof. **<u>Exercise</u>.** $\square$

Proposition 2.30. *Let $(S, \mathcal{M}, \mu)$ be a measure space and $f : S \to \mathbb{K}$ measurable with respect to $\mathcal{M}^*$. Then, there exists a function $g : S \to \mathbb{K}$ such that g is measurable with respect to $\mathcal{M}$ and g does not differ from f outside of a subset $N \in \mathcal{M}$ of measure 0.*

Proof. By Theorem 2.23 there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ of simple maps with respect to $\mathcal{M}^*$ that converges pointwise to f. For each f_n we can find a set $N_n \in \mathcal{M}$ of measure 0 such that the function $k_n : S \to \mathbb{K}$ defined by $k_n(p) = f_n(p)$ if $p \in S \setminus N_n$ and $k_n(p) = 0$ otherwise, is simple with respect to $\mathcal{M}$. (**<u>Exercise</u>.**Show this!) The set $N := \bigcup_{n=1}^{\infty} N_n \in \mathcal{M}$ has measure zero. Moreover, $g_n : S \to \mathbb{K}$ defined by $g_n(p) = f_n(p)$ if $p \in S \setminus N$ and $g_n(p) = 0$ otherwise, is simple with respect to $\mathcal{M}$. Moreover, the sequence $\{g_n\}_{n\in\mathbb{N}}$ converges pointwise to $g : S \to \mathbb{K}$ defined by $g(p) = f(p)$ if $p \in S \setminus N$ and $g(p) = 0$ otherwise. Thus, by Theorem 2.19, g is measurable with respect to $\mathcal{M}$. $\square$

2.4 Extension of Measures

We now turn to the question of how to construct measures. We will focus here on the method of extension. That is, we consider a measure that is merely defined on an algebra of subsets and extend it to a measure on a σ-algebra.

Definition 2.31. Let S be a set and $\mathcal{M}$ a σ-algebra of subsets of S. Then, a map $\lambda : \mathcal{M} \to [0, \infty]$ is called an *outer measure* on $\mathcal{M}$ iff it satisfies the following properties:

- $\lambda(\emptyset) = 0$.
- Let $A, B \in \mathcal{M}$ and $A \subseteq B$. Then, $\lambda(A) \le \lambda(B)$. (monotonicity)
- Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of elements of $\mathcal{M}$. Then,

$$\lambda\left(\bigcup_{n\in\mathbb{N}} A_n\right) \le \sum_{n\in\mathbb{N}} \lambda(A_n). \quad \text{(countable subadditivity)}$$

Lemma 2.32. *Let S be a set, $\mathcal{N}$ an algebra of subsets of S and μ a measure on $\mathcal{N}$. On the σ-algebra $\mathfrak{P}(S)$ of all subsets of S define the function $\lambda : \mathfrak{P}(S) \to [0, \infty]$ given by*

$$\lambda(X) = \inf\left\{\sum_{n\in\mathbb{N}} \mu(A_n) : A_n \in \mathcal{N}\, \forall n \in \mathbb{N} \, and \, X \subseteq \bigcup_{n\in\mathbb{N}} A_n\right\}.$$

Then, λ is an outer measure on $\mathfrak{P}(S)$. Moreover, it extends μ, i.e., $\lambda(A) = \mu(A)$ for all $A \in \mathcal{N}$.

Proof. **Exercise.** □

Definition 2.33. Let S be a set and λ an outer measure on the σ-algebra $\mathfrak{P}(S)$ of all subsets of S. Then, $A \subseteq S$ is called *λ-measurable* iff $\lambda(X) = \lambda(X \cap A) + \lambda(X \cap \neg A)$ for all $X \subseteq S$.

Lemma 2.34. *Let S be a set and λ an outer measure on the σ-algebra $\mathfrak{P}(S)$ of all subsets of S. Let $\mathcal{M}$ be the set of subsets of S that are λ-measurable. Then, $\mathcal{M}$ is a σ-algebra and λ is a complete measure on $\mathcal{M}$.*

Proof. **Exercise.** □

Theorem 2.35 (Hahn)**.** *Let S be a set, $\mathcal{N}$ an algebra of subsets of S and μ a measure on $\mathcal{N}$. Then, μ can be extended to a σ-algebra $\mathcal{M}$ containing $\mathcal{N}$ such that μ is a complete measure on $\mathcal{M}$ and for all $X \in \mathcal{M}$ we have*

$$\mu(X) = \inf \left\{ \sum_{n\in\mathbb{N}} \mu(A_n) : A_n \in \mathcal{N}\, \forall n \in \mathbb{N} \, \mathit{and}\, X \subseteq \bigcup_{n\in\mathbb{N}} A_n \right\}.$$

Proof. **Exercise.** □

Proposition 2.36 (Uniqueness of Extension)**.** *Let S be a measurable space with σ-algebra $\mathcal{M}$ and measures μ_1, μ_2. Suppose there is an algebra $\mathcal{N} \subseteq \mathcal{M}$ generating $\mathcal{M}$ and such that $\mu(A) := \mu_1(A) = \mu_2(A)$ for all $A \in \mathcal{N}$. Furthermore, assume that μ is σ-finite with respect to $\mathcal{N}$. Then, $\mu_1 = \mu_2$ also on $\mathcal{M}$.*

Proof. Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of elements of $\mathcal{N}$ such that $S = \bigcup_{n\in\mathbb{N}} X_k$ and $X_n \subseteq X_{n+1}$ and $\mu(X_n) < \infty$ for all $n \in \mathbb{N}$. (By σ-finiteness, there is a sequence $\{Y_k\}_{k\in\mathbb{N}}$ with $S = \bigcup_{k\in\mathbb{N}} Y_k$ and $\mu(Y_k) < \infty$ for all $k \in \mathbb{N}$. Now set $X_n := \bigcup_{k=1}^{n} Y_k$.) Define the finite measures $\mu_{1,n}(A) := \mu_1(A \cap X_n)$ and $\mu_{2,n}(A) := \mu_2(A \cap X_n)$ on $\mathcal{M}$ for all $n \in \mathbb{N}$. Now, let $\mathcal{B}_n$ be the subsets of $\mathcal{M}$ where $\mu_{1,n}$ and $\mu_{2,n}$ agree. By construction, $\mathcal{N} \subseteq \mathcal{B}_n$ for all $n \in \mathbb{N}$. We show that the $\mathcal{B}_n$ are monotone.

Fix $n \in \mathbb{N}$. Let $\{A_k\}_{k\in\mathbb{N}}$ be a sequence of elements of $\mathcal{B}_n$ such that $A_k \subseteq A_{k+1}$ for all $k \in \mathbb{N}$ and set $A := \bigcup_{k\in\mathbb{N}} A_k$. Then, using Proposition 2.26,

$$\mu_{1,n}(A) = \lim_{k\to\infty} \mu_{1,n}(A_k) = \lim_{k\to\infty} \mu_{2,n}(A_k) = \mu_{2,n}(A).$$

So, $A \in \mathcal{B}_n$. Now, let $\{A_k\}_{k\in\mathbb{N}}$ be a sequence of elements of $\mathcal{B}_n$ such that $A_k \supseteq A_{k+1}$ for all $k \in \mathbb{N}$ and set $A := \bigcap_{k\in\mathbb{N}} A_k$. Again using Proposition 2.26 we get (note that the finiteness of the measure is essential here),

$$\mu_{1,n}(A) = \lim_{k\to\infty} \mu_{1,n}(A_k) = \lim_{k\to\infty} \mu_{2,n}(A_k) = \mu_{2,n}(A).$$

So, $A \in \mathcal{B}_n$. Hence, $\mathcal{B}_n$ is monotone and by Proposition 2.9 we must have $\mathcal{M} \subseteq \mathcal{B}_n$ and hence $\mathcal{M} = \mathcal{B}_n$.

Thus, $\mu_{1,n} = \mu_{2,n}$ for all $n \in \mathbb{N}$. But then, $\mu_1 = \lim_{n\to\infty} \mu_{1,n} = \lim_{n\to\infty} \mu_{2,n} = \mu_2$. This completes the proof. □

Proposition 2.37. *Let $(S, \mathcal{M}, \mu)$ be a measure space. Let N be an algebra of subsets of S that generates $\mathcal{M}$. Denote the completion of $\mathcal{M}$ with respect to μ by $\mathcal{M}^*$. Then, for any $X \in \mathcal{M}^*$ with finite measure and any $\epsilon > 0$ there exists $A \in \mathcal{N}$ such that*

$$\mu((X \setminus A) \cup (A \setminus X)) < \epsilon.$$

Proof. Let $X \in \mathcal{M}^*$. By Hahn's Theorem 2.35 there exists a sequence $\{A_n\}_{n\in\mathbb{N}}$ of disjoint elements of $\mathcal{N}$ such that $X \subseteq \bigcup_{n\in\mathbb{N}} A_n$ and

$$\sum_{n=1}^{\infty} \mu(A_n) < \mu(X) + \epsilon/2.$$

Now fix $k \in \mathbb{N}$ such that

$$\sum_{n=k+1}^{\infty} \mu(A_n) < \epsilon/2.$$

Set $A := \bigcup_{n=1}^{k} A_n$. Then, on the one hand,

$$\mu(A \setminus X) \leq \mu\left(\left(\bigcup_{n=1}^{\infty} A_n\right) \setminus X\right) < \epsilon/2,$$

while on the other hand,

$$\mu(X \setminus A) \leq \mu\left(\left(\bigcup_{n=1}^{\infty} A_n\right) \setminus A\right) = \mu\left(\bigcup_{n=k+1}^{\infty} A_n\right) < \epsilon/2.$$

This implies the statement. □

2.5 The Lebesgue Measure

In the following we are going to construct the Lebesgue measure. This is the unique (as we shall see) measure on the real numbers assigning to an interval its length. The construction proceeds in various stages.

Lemma 2.38. *The finite unions of intervals of the type $[a, b)$, $(-\infty, a)$, and $[a, \infty)$ together with $\emptyset$ form an algebra $\mathcal{N}$ of subsets of the real numbers.*

Proof. **Exercise.** □

Lemma 2.39. *The prescription $\mu([a, b)) = b - a$ determines uniquely a finitely additive function $\mu : \mathcal{N} \to [0, \infty]$ on the algebra $\mathcal{N}$ considered above.*

Proof. **Exercise.** □

Lemma 2.40. *The function $\mu : \mathcal{N} \to [0, \infty]$ defined above is countably additive and thus a measure.*

Proof. Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of pairwise disjoint elements of $\mathcal{N}$ such that $A := \bigcup_{n\in\mathbb{N}} \in \mathcal{N}$. We wish to show that

$$\mu(A) = \sum_{n\in\mathbb{N}} \mu(A_n).$$

By finite additivity we have $\mu(A) \geq \mu(\bigcup_{n=1}^{m} A_n) = \sum_{n=1}^{m} \mu(A_n)$ for all $m \in \mathbb{N}$ and hence

$$\mu(A) \geq \sum_{n\in\mathbb{N}} \mu(A_n).$$

It remains to show the opposite inequality.

Assume at first that A is a finite interval $[a, b)$. Then, A is the disjoint union of a sequence of intervals $\{I_k\}_{k\in\mathbb{N}}$ with $I_k = [a_k, b_k)$ in such a way that each A_n is the finite union of some I_k. (We also allow the degenerate case $a_k = b_k$ in which case $I_k = \emptyset$.) Fix now $\epsilon > 0$ (with $\epsilon < b - a$) and define $I'_k := (a_k - 2^{-(k+1)}\epsilon, b_k)$ for all $k \in \mathbb{N}$. Then, the open sets $\{I'_k\}_{k\in\mathbb{N}}$ cover the compact interval $[a, b - \epsilon/2]$. Thus, there is a finite set of indices $I \subset \mathbb{N}$ such that $[a, b - \epsilon/2] \subset \bigcup_{k\in I} I'_k$. Then clearly also $[a, b - \epsilon/2) \subset \bigcup_{k\in I} I''_k$, where $I''_k := [a_k - 2^{-(k+1)}\epsilon, b_k)$. By finite additivity of μ we get

$$\begin{aligned}\mu([a, b - \epsilon/2)) \leq \mu\left(\bigcup_{k\in I} I''_k\right) &\leq \sum_{k\in I} \mu\left(I''_k\right) \\ &= \sum_{k\in I} \left(\mu(I_k) + \epsilon 2^{-(k+1)}\right) \leq \epsilon/2 + \sum_{k\in I} \mu(I_k).\end{aligned}$$

But since $\mu(A) = \mu([a, b-\epsilon/2)) + \epsilon/2$, we find $\mu(A) \leq \epsilon + \sum_{k\in I} \mu(I_k)$. Thus, there exists $m \in \mathbb{N}$ such that $\mu(A) \leq \epsilon + \sum_{n=1}^{m} \mu(A_n)$. But since ϵ was arbitrary we can conclude $\mu(A) \leq \sum_{n\in\mathbb{N}} \mu(A_n)$ and hence equality.

Exercise. Complete the proof. □

Proposition 2.41. *Consider the real numbers with its σ-algebra $\mathcal{B}$ of Borel sets. Then, the prescription $\mu([a, b)) := b - a$ uniquely extends to a measure $\mu : \mathcal{B} \to [0, \infty]$.*

Proof. By Lemmas 2.38, 2.39 and 2.40 the prescription uniquely defines a measure μ on the algebra $\mathcal{N}$ of unions of intervals of the type $[a, b)$, $(-\infty, a)$, and $[a, \infty)$. By Theorem 2.35 μ extends to a σ-algebra $\mathcal{M}$ containing $\mathcal{N}$. But the σ-algebra generated by $\mathcal{N}$ is the σ-algebra $\mathcal{B}$ of Borel sets. (**Exercise.**Show this!) So, in particular, we get a measure on $\mathcal{B}$. By Proposition 2.36 this is unique since μ is σ-finite on $\mathcal{N}$. (**Exercise.**Show this latter statement!) □

Definition 2.42. The measure defined in the preceding Proposition is called the *Lebesgue measure* on $\mathbb{R}$.

Exercise 16. Consider the real numbers with the Lebesgue measure. Determine $\mu(\mathbb{Q})$ and $\mu(\mathbb{R} \setminus \mathbb{Q})$.

Exercise 17. The Cantor set C is a subset of the interval $[0,1]$. It can be described for example as

$$C = \bigcap_{n=0}^{\infty} \bigcup_{k=0}^{(3^n-1)/2} \left[\frac{2k}{3^n}, \frac{2k+1}{3^n}\right].$$

Show that $\mu(C) = 0$.

Proposition 2.43. *The Lebesgue measure is translation invariant, i.e., $\mu(A+c) = \mu(A)$ for any measurable A and $c \in \mathbb{R}$.*

Proof. Straightforward. □

Exercise 18. Consider the following equivalence relation on $\mathbb{R}$: Let $x \sim y$ iff $x - y \in \mathbb{Q}$. Now choose (using the axiom of choice) one representative out of each equivalence class, such that this representative lies in $[0,1]$. Call the set obtained in this way A.

1. Show that $(A+r) \cap (A+s) = \emptyset$ if r and s are distinct rational numbers. Supposing that A is Lebesgue measurable, conclude that $\mu(A) = 0$.

2. Show that $\mathbb{R} = \bigcup_{q \in \mathbb{Q}} (A+q)$. Supposing that A is Lebesgue measurable, conclude that $\mu(A) > 0$.

We obtain a contradiction showing that A is not Lebesgue measurable.

We can define the Lebesgue measure more generally for $\mathbb{R}^n$. The intervals of the type $[a,b)$ are replaced by products of such intervals. Otherwise the construction proceeds in parallel.

Proposition 2.44. *Consider $\mathbb{R}^n$ with its σ-algebra $\mathcal{B}$ of Borel sets. Then, the prescription $\mu([a_1,b_1) \times \cdots \times [a_n,b_n)) = (b_1 - a_1) \cdots (b_n - a_n)$ uniquely extends to a measure $\mu : \mathcal{B} \to [0,\infty]$.*

Exercise 19. Sketch the proof by explaining the changes with respect to the one-dimensional case.